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**PREFACE**

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# ON THE FOUNDATIONS OF RELATIVISTIC ENERGY MECHANICS

by

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## SUMMARY

The Einstein theory of general relativity is shown to yield a general mechanics of continuous media under the assumption that the momentum-energy tensor admits a unique time-like eigenvector. Physical interpretations of the governing equations are derived, together with constitutive relations for general and isotropic materials. It turns out that the mechanics can always be viewed as describing the flow of rest-energy. Invariant requirements for the existence of a stress potential are obtained, the satisfaction of which leads to a decomposition and partial evaluation of the rest-energy. The Einstein field equations are shown to imply the existence and uniqueness of an intrinsic energy density for any material medium (intrinsic immutable mass). The usual procedure of adding conditions to the Einstein theory in order to obtain an analogous intrinsic quantity is thus unnecessary. The path density, which defines the intrinsic energy, is shown to be path independent in an appropriate sense if the stresses admit a stress potential. This suggests a decomposition of the generalized stresses and leads to a fundamental differential relation on the trajectories of the energy flux. Natural definitions of intrinsic temperature and intrinsic entropy density are direct consequences of the fundamental differential relation and lead to generalized thermodynamic descriptions which include the effects of gravitational radiation.

## 1. INTRODUCTION

Since the promulgation of the general theory of relativity, science has come to view the problem of describing general media as equivalent to the problem of stating the momentum-energy tensor for such media. For certain special cases, the appropriate forms for the momentum-energy tensor are known. These have been most useful in gleaning the dynamic interplay of geometry and physics. Unfortunately, without significant extension, these cases regulate the investigator to the consideration of only the simplest of possible physical models. In view of the fundamental nature of the Einstein theory, it should be possible to develop a simple and consistent mechanics of general systems which involves a minimum of assumptions and which yields direct physical interpretations of all quantities involved. The purpose of this paper is to lay the requisite foundations for such a mechanics.

## 2. THE MOMENTUM-ENERGY TENSOR OF A MATERIAL MEDIUM

Let  $\mathcal{E}$  be an Einstein-Riemann space\* with metric tensor  $h_{AB}$  and associated momentum-energy tensor  $T_{AB}$ . Capital Latin indices are assumed to range through the integers from one to four. A semicolon will be used to denote covariant differentiation, the comma being reserved to denote partial differentiation. The usual notation for mixing and alternating will be used.

Following the line of thought introduced by Synge\*\*, it appears natural, and indeed necessary, to require that any region of  $\mathcal{E}$  which is filled by a

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\* A four-dimensional metric space of the hyperbolic-normal type in which Einstein's field equations are satisfied. The signature of  $\mathcal{E}$  is therefore -2.

Synge, J. L., Relativity: The General Theory, North-Holland Publishing Co., Amsterdam (1960), pp. 168-169.

material medium be such that its associated momentum-energy tensor admit a unique time-like eigenvector. The precise characterization is as follows:

**DEFINITION 1.** A symmetric tensor field  $T_{AB}$  is a material momentum-energy tensor associated with a region  $\mathcal{B}$  of  $\mathcal{E}$  if and only if

$$(2.1) \quad T^{AB}_{;B} = 0$$

and there exists a unique, time-like, unit vector field  $W^A$  and a nonzero scalar  $\mu$  on  $\mathcal{B}$  such that

$$(2.2) \quad T^A_B W^A = \mu W^B.$$

Careful attention should be paid to the requirement that the vector field  $W^A$  be unique. This uniqueness states that any point  $P$  of the region  $\mathcal{B}$  has associated with it a unique time-direction defined by  $W^A$ , and hence the material medium filling  $\mathcal{B}$  has a uniquely defined time-orientation. Without such a result, it is difficult, if not altogether impossible—to speak meaningfully of the simplest everyday properties of what we are accustomed to think of as material bodies. Care must also be exercised in the use of Def. 1, for we have to distinguish between the various cases which may arise.

**DEFINITION 2.** A material momentum-energy tensor is said to be of class  $m$  ( $m = 1, 2, 3, 4$ ) if and only if the multiplicity of the eigenvalue  $\mu$  is  $m$ .

In order to guarantee the unicity of the eigenvector  $W^A$ , material momentum-energy tensors of class  $m$  must be such that their characteristics (referred to Jordan normal form) commence with the integer  $m$ .<sup>\*</sup> Since space-like and

<sup>\*</sup> For a detailed discussion of such characteristics see Hlavatý, V., "Contribution to the Theory of General Geometrodynamics" (to be published).

time-like quantities usually differ markedly in their numerical values when calculated with other than what may be termed exotic units, it is natural to expect that material momentum-energy tensors of class one predominate. For this reason, we confine our attention in the remainder of this paper to such momentum-energy tensors.

### 3. CHARACTERIZATION OF MATERIAL MOMENTUM-ENERGY TENSORS OF CLASS ONE

The definitions given in the previous section are implicit, for they do not exhibit the detailed form and nature of  $T^{AB}$ . This section is devoted to developing the explicit characterization which is needed in order to obtain physical interpretations.

Consider the idempotent tensor field

$$(3.1) \quad \pi_A^B = \delta_A^B - W_A W^B$$

on  $\mathcal{B}$ . With it we may define a projection operator  $\pi$  over the set of all geometric objects on  $\mathcal{B}$  as follows:

$$\pi(I_{B \dots}^{A \dots}) = \pi_C^A \dots \pi_B^D \dots I_{D \dots}^{C \dots}.$$

In particular, we have

$$(3.2) \quad \pi(W_{A;B}) = W_{(A;B)} - W_{(A} \dot{W}_{B)} + W_{[A;B]} + W_{[A} \dot{W}_{B]},$$

where

$$(3.3) \quad \dot{W}_A \stackrel{\text{def}}{=} W_{A;B} W^B.$$

(Here we have used the fact that  $W^A W_A = 1$  implies  $W^A W_{B;C} = 0$ .) A direct application of well known techniques together with definitions 1 and 2 then yields the following result. The most general symmetric tensor field of class one which satisfies (2.2) is given by

$$(3.4) \quad T^{AB} = \mu W^A W^B + \sigma^{AB},$$

where  $\sigma^{AB}$  is any symmetric tensor field such that

$$(3.5) \quad \pi(\sigma^{AB}) = \sigma^{AB}$$

and

$$(3.6) \quad \sigma^{AB} V_B = \mu V^A$$

if and only if  $V_B = 0$ .

With the above result, all that remains in order for  $T^{AB}$  to be material is satisfaction of equations (2.1). Substituting (3.4) into (2.1) gives

$$(3.7) \quad \mu \dot{W}^A + W_A (\mu W^B)_{;B} + \sigma^{AB}_{;B} = 0.$$

Now, by assumption we have  $W^A W_A = 1$ , while (3.5) implies that  $\sigma^{AB} W_B = 0$ .

It thus follows that  $W_A \dot{W}^A = 0$  and

$$\sigma^{AB}_{;B} W_A = -\sigma^{AB} W_{A;B} = -\sigma^{AB} W_{(A;B)}$$

since  $\sigma^{AB}$  is symmetric. Noting that  $\sigma^{AB} W_{(A;B)}$  is a scalar and that  $\sigma^{AB}$  satisfies (3.5), we have

$$(3.8) \quad \sigma^{AB}_{;B} W_A = -\sigma^{AB} \epsilon_{AB},$$

where (vid 3.2)

$$(3.9) \quad \epsilon_{AB} \stackrel{\text{def}}{=} \pi(W_{A;B}) = W_{(A;B)} - W_A \dot{W}_B$$

is the Born rate-of-strain tensor associated with the vector field  $W^A$ .\*

Hence, if we contract (3.7) with  $W_A$  we are led to the scalar equation

$$(3.10) \quad (\mu W^B)_{;B} = \sigma^{AB} \epsilon_{AB}.$$

With the aid of the Lie derivative,\*\* we can write (3.10) in the more suggestive

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\* Sygne; *op. cit.*, p. 172, eq. 62. The difference in sign between our statement and that of Sygne arises because Sygne assumes a signature +2 while we have assumed a signature -2. The physical interpretation of  $\epsilon_{AB}$  is discussed in Sec. 7.

\*\* Yano, K., *The Theory of Lie Derivatives and Its Applications*, North-Holland Publishing Co., Amsterdam (1957).



form

$$(3.11) \quad \mathcal{L}(h\mu) = h\sigma^{AB}\epsilon_{AB}$$

where

$$h \stackrel{\text{def}}{=} (-\det(h_{AB}))^{1/2}.$$

Substituting (3.20) back into (3.7) then leads to

$$(3.12) \quad \mu \dot{W}^A + W^A \sigma^{BC}\epsilon_{BC} + \sigma^{AB}{}_{;B} = 0.$$

We have thus established the following result. The most general material momentum-energy tensor of class one is given by

$$(3.13) \quad T^{AB} = \mu W^A W^B + \sigma^{AB}$$

where  $\sigma^{AB}$  is a symmetric tensor such that

$$(3.14) \quad \pi(\sigma^{AB}) = \sigma^{AB},$$

$$(3.15) \quad \sigma^{AB} V_B = \mu V^A$$

if and only if  $V_B = 0$ , and the quantities  $\mu$ ,  $W^A$ , and  $\sigma^{AB}$  are such that

$$(3.16) \quad \mathcal{L}(h\mu) = h\sigma^{AB}\epsilon_{AB},$$

$$(3.17) \quad \mu \dot{W}^A + W^A \sigma^{BC}\epsilon_{BC} + \sigma^{AB}{}_{;B} = 0.$$

#### 4. PHYSICAL INTERPRETATIONS

The question naturally arises as to just what physical interpretations can be attached to the quantities  $\mu$ ,  $W^A$ , and  $\sigma^{AB}$ . We take here the viewpoint that physical interpretations should be consequences of the mathematical model employed.

We first show that the vector field  $W^A$  may be interpreted as a velocity field. Since  $W^A$  is assumed to be known throughout the region  $\mathcal{B}^*$  we may define a system of fibers in  $\mathcal{B}$  by

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$$^*W^A \text{ is uniquely determined by } T^{AB} \text{ and } W^A W_A = 1.$$

$$(4.1) \quad dx^A/d\eta = W^A(x)$$

where  $\eta$  is the fiber parameter. Computing the differential element of arc length along any fiber in  $\mathcal{L}$ , we are leads to

$$ds^2 = h_{AB} W^A W^B d\eta^2 = d\eta^2.$$

Hence (4.1) shows that  $W^A$  satisfies the definition of a velocity field in general relativity.

The next question is what physical quantity admits  $W^A$  as its velocity field. From (2.2) we obtain

$$(4.2) \quad \mu = T_{AB} W^A W^B.$$

Transforming to a frame of reference in  $\mathcal{L}^*$  such that\*

$$(4.3) \quad W^A = \delta_0^A,$$

equation (4.2) reduces to  $\mu = T_{00}$ . Thus, since (4.2) defines a rest frame relative to the velocity field  $W^A$ , and the  $T_{00}$  appearing above is the energy in this rest frame,\*\* the scalar character of  $\mu$  shows that  $\mu$  is the rest-energy in all frames of reference. Writing (3.16) in the equivalent form

$$(\mu h W^A)_{,A} = h \sigma^{AB} \epsilon_{AB}$$

shows that  $\mu h W^A$  is a flux of rest-energy density that is created in an amount equal to  $\sigma^{AB}_{AB}$  per unit of geometrical volume  $h dx^1 dx^2 dx^3 dx^4$ . From the obvious analogy with fluid mechanics, equations (3.17) may be interpreted as

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\* This can always be done for any vector field with the properties exhibited by  $W^A$ : see E. Goursat, Leçons sur le problème de Pfaff, Herman et Cie., Paris (1922), p. 117. The symbol  $*$  will be used to denote evaluation in this frame of reference.

\*\* From (4.3) and  $W_A W^A = 1$  we obtain  $h_{00} = 1$ . Hence,  $T_{00}$  in the frame defined by (4.3) is numerically equal to the Minkowskian value of  $T_{00}$ —the energy.

describing the motion of a fluid with mass density  $\mu$  and stress tensor  $\sigma^{AB}$ , where the mass density is created in conformity with equation (3.16). The interpretation of  $\sigma^{AB}$  as a generalized stress is also operationally admissible. This follows from the fact that, by (3.14), the support of  $\sigma^{AB}$  is the generic three-dimensional space  $\mathcal{U}$  orthogonal to  $W^A$ ,\* and hence  $\sigma^{AB}$  may be determined in the frame  $W^A = \delta_0^A$  by only space-like measurements. Thus, since the quantity  $\mu$  has been shown to be the rest-energy, we may interpret the vector field  $W^A$  as the velocity field associated with the flow of rest-energy in the region  $\mathcal{B}$  and  $\sigma^{AB}$  as the generalized stress that gives rise to this energy flow.

The above interpretations have been drawn in a manner which is independent of what actual physical processes occur in  $\mathcal{B}$ ; the only proviso being that  $T^{AB}$  be a material momentum-energy tensor of class one. We may thus view the dynamical processes in any region  $\mathcal{B}$  with a material momentum-energy tensor as the flow of rest-energy regardless of what actual physical processes occur. The continuum in relativity theory thus turns out to be an energy continuum—thus the name relativistic energy mechanics.

## 5. BOUNDARY CONDITIONS

The next problem to be resolved in the resultant energy mechanics is that of formulating consistent boundary conditions. Let  $\mathcal{B}$  be a region of  $\mathcal{E}$  which supports an energy mechanics (i. e., the associated momentum-energy tensor of  $\mathcal{B}$  is material and of class one), and assume that exterior to  $\mathcal{B}$ ,  $T^{AB}$

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\*The generic space  $\mathcal{U}$  becomes a proper space if and only if  $\pi(W_{[A;B]}) = 0$ : see D. G. B. Edelen, "Rotation Tensors and Irrotational Motions in Einstein-Riemann Spaces," Proc. Nat. Acad. Sci. (1963) (to be published).

is not material. If we denote the boundary of  $\mathcal{B}$  by  $\partial\mathcal{B}$  and the outward normal to  $\partial\mathcal{B}$  by  $N^B$ , then a necessary condition\* for the solution of the Einstein field equations is

$$(5.1) \quad [T^{AB}] N_B = 0, \quad ([T^{AB}] \stackrel{\text{def}}{=} T^{AB} - \bar{T}^{AB}).$$

Here  $T^{AB}$  ( $\bar{T}^{AB}$ ) denotes the value of  $T^{AB}$  just exterior (interior) to  $\partial\mathcal{B}$ .

Under the above assumptions, we may use (3.13) to evaluate  $\bar{T}^{AB}$ . When this is substituted into (6.1), we obtain the conditions

$$(5.2) \quad (\bar{\mu} \bar{W}^A \bar{W}^B + \bar{\sigma}^{AB}) N_B = T^{AB} N_B.$$

Hence, if we operate on both sides of (6.2) with  $\pi$ , set

$$(5.3) \quad F^A = \pi(T^{AB} N_B),$$

and use the projective invariance of  $\sigma^{AB}$  as stated by (3.14), we are led to

$$(5.4) \quad \bar{\sigma}^{AB} \pi(N_B) = F^A.$$

Equations (6.4) are just the relativistic analog of the classical stress boundary conditions, as is easily seen by referring (5.4) to a rest frame relative to  $W^A$ .

Combining (5.2) and (5.4), the corresponding boundary conditions of the  $W^A$  are easily obtained; namely

$$(5.5) \quad \bar{\mu} (\bar{W}^B N_B) \bar{W}^A = T^{AB} N_B - F^A.$$

Hence, since  $\bar{\mu} \neq 0$ , a necessary and sufficient condition for  $\bar{W}^A$  to be tangent to  $\partial\mathcal{B}$  is

$$N_A (1 - \pi)(T^{AB} N_B) = 0.$$

The evaluation of the corresponding conditions for the  $h_{AB}$ 's and the resulting

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\*Edelen, D. G. B., and T. Y. Thomas, "Discontinuities in the Einstein Field for General Momentum-Energy Tensors," Arch. Rational Mech. Anal. 9 (1962), pp. 153-171.

surface dynamics has been treated elsewhere, \*

## 6. CONSTITUTIVE RELATIONS

The relativistic energy mechanics established in the previous sections lacks constitutive relations whereby the generalized stresses  $\sigma^{AB}$  are related to the velocity, the velocity gradient, and any other fields which may be present. Such relations are a necessary part of the theory, for without them the Einstein field equations and the basic equations of relativistic energy mechanics would be insufficient to determine a tensorial solution manifold. \*

DEFINITION 3. A relation of the form

$$(6.1) \quad \sigma^{AB} = Q^{AB}(w_C, w_{E;F}, h_{KL}, \dots)$$

is said to be a constitutive relation in the relativistic energy mechanics.

The point of departure in constructing such relations is the data supplied by section 3; namely,  $\sigma^{AB}$  is symmetric and such that

$$(6.2) \quad \pi(\sigma^{AB}) = \sigma^{AB},$$

$$(6.3) \quad \sigma^{AB} v_B = \mu v^A$$

if and only if  $v_B = 0$ . Since (6.2) requires the support of  $\sigma^{AB}$  to be the generic three-dimensional space  $\mathcal{U}$  orthogonal to the vector field  $w^A$ , it would appear natural to confine our attention to quantities whose support is also  $\mathcal{U}$ .

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\* Edelen, D. G. B., and T. Y. Thomas, "The Dynamics of Discontinuity Surfaces in General Relativity," (to be published).

\*\* The Einstein field equations together with basic equations of energy mechanics constitute a system of 11 independent equations for the determination of the variables  $h_{AB}$ ,  $\mu$ ,  $w^A$  and  $\sigma^{AB}$  among which there are only 25 independent ones. Thus, if we append ten equations for the determination of the ten  $\sigma^{AB}$ 's without introducing any new variables the solution manifold would be tensorial because of the occurrence of four arbitrary functions of position, the latter being determined by the choice of the coordinate system.

Physically, this would mean that an observer in a frame of reference at rest with respect to the velocity field  $W^A$  could determine both constitutive relations and stresses by measurements of only space-like quantities. This, however, is exactly the state of affairs in the classical mechanics of deformable continua.

We are therefore led to the following definition.

**DEFINITION 4.** A relation of the form

$$(6.4) \quad \sigma^{AB} = C^{AB}(\tau(W_{A;B}), \tau(h_{AB}), \dots)$$

is said to be a classically admissible constitutive relation in the relativistic energy mechanics, and the resulting energy mechanics is said to be classically admissible.

It is to be expected that only in the cases of classically admissible constitutive relations will the relativistic energy mechanics reduce to classical continuum mechanics by an appropriate limiting process.

The simplest kinds of classical continua are those referred to as isotropic. Although the idea of isotropy is well defined in the classical theory, there are several possible formulations available in the space-time of general relativity. The simplest and most obvious intuitively is that associated with the idea of coincidence of the principal directions of stress and the velocity gradients. A necessary and sufficient condition for coincidence of the principal axes of  $\sigma_{AB}$  and  $W_{A;B}$  is that  $\sigma_{AB}$  be a power series in  $W_{A;B}$  and  $W_{B;A}$ , with scalar functions of  $W_{A;B}$  as coefficients. This would violate the symmetry of  $\sigma^{AB}$ . Hence we can at most require that the principal axes of  $\sigma_{AB}$  and  $W_{(A;B)}$  coincide. Applying the same reasoning in conjunction with the condition (5.2) shows that we

can at most require coincidence of the principal axes of  $\sigma_{AB}$  and  $\pi(W_{(A;B)})$ . Hence we are led to the following definition of isotropy.

**DEFINITION 5.** A relativistic energy continuum is isotropic if and only if the axes of  $\sigma_{AB}$  coincide with the axes of  $\epsilon_{AB}$ .

If we set

$$(6.5) \quad \epsilon_{AB}^B = \epsilon_{AC} K^{CB}, \quad \epsilon_{AB}^{AB} = \epsilon_{CD} k^{CA} k^{DB},$$

where

$$(6.6) \quad k^{AB} = \pi(h^{AB}), \quad k_{AB} = \pi(h_{AB}),$$

then the most general constitutive relations for an isotropic continuum which satisfy the conditions  $\sigma^{[AB]} = 0$  and (6.2) are given by

$$(6.7) \quad \sigma^{AB} = \psi_0 k^{AB} + \psi_1 \epsilon^{AB} + \psi_2 \epsilon^{AC} \epsilon^B_C + \dots$$

where the coefficients  $\psi_\Lambda$  are scalar functions of  $W_{A;B}$ . Since  $\det(\epsilon_{AB}) = 0$

and  $\epsilon_{AB}$  satisfies its own characteristic equation, we have

$$(6.8) \quad \epsilon^{AC} \epsilon^D_C \epsilon^B_D + I_1 \epsilon^{AC} \epsilon^B_C + I_2 \epsilon^{AB} - I_3 = 0,$$

where

$$(6.9) \quad \begin{cases} I_1 = \epsilon^A_A \\ I_2 = \frac{1}{4!} h^{-2} \epsilon^{ABCD} \epsilon^{FGHK} h_{AF} h_{BG} \epsilon^{CH} \epsilon^{DK} \\ I_3 = \frac{1}{4!} h^{-2} \epsilon^{ABCD} \epsilon^{FGHK} h_{AF} \epsilon^{BG} \epsilon^{CH} \epsilon^{DK} \end{cases}$$

and  $\epsilon^{ABCD}$  is the Levi-Civita indicator. Hence, since the  $W_{A;B}$  dependence of the scalar functions  $\psi_\Lambda$  can be written in terms of the invariants

$$(6.10) \quad \begin{cases} \Pi_1 = W^A_{A;A} \\ \Pi_2 = \frac{1}{4!} h^{-2} \epsilon^{ABCD} \epsilon^{FGHK} h_{AF} h_{BG} W_{C;H} W_{D;K} \\ \Pi_3 = \frac{1}{4!} h^{-2} \epsilon^{ABCD} \epsilon^{FGHK} h_{AF} W_{B;G} W_{C;H} W_{D;K} \\ \Pi_4 = h^{-2} \det(W_{A;B}), \end{cases}$$

the most general isotropic energy continuum admits the constitutive relations

$$(6.11) \quad \sigma^{AB} = \varphi_0 k^{AB} + \varphi_1 \epsilon^{AB} + \varphi_2 \epsilon^{AC} \epsilon^B_C,$$

where

$$(6.12) \quad \varphi_\Lambda = f_\Lambda (I_1, I_2, I_3, II_1, II_2, II_3, II_4, \dots), \quad \Lambda = 0, 1, 2,$$

and the  $f$ 's are scalar functions of their indicated arguments. If we require the continuum to be classically admissible, then  $\sigma^{AB}$  is again given by (6.11), but the  $f_\Lambda$  are no longer functions of the  $I$ 's, and the  $h$ 's appearing in (6.9) are replaced by  $k$ 's.

We still have to satisfy the requirement (6.3). Let  $e_i$  ( $i = 1, 2, 3$ ) be defined by

$$(6.13) \quad \epsilon^{AB} v_B = e_i v_i^A, \quad (i \text{ not summed}),$$

where  $\pi(v_A) = v_A$ . Since the projective invariance of the  $V$ 's implies  $v_A k^{AB} = v^B$ , (6.11) gives

$$(6.14) \quad \sigma^{AB} v_B = (\varphi_0 + \varphi_1 e_i + \varphi_2 e_i^2) v_i^A \quad (i \text{ not summed}).$$

Hence we must require that the  $\varphi$ 's and the  $e_i$  be such that

$$(6.15) \quad \varphi_0 + \varphi_1 e_i + \varphi_2 e_i^2 \neq 0.$$

If the above results are examined in a frame of reference at rest with respect to the velocity field  $W^A$ , the classically admissible constitutive relations are easily seen to be exactly the results known in the classical theory.

As an example, consider the relation

$$(6.16) \quad \sigma^{AB} = (p + \lambda \epsilon^C_C) k^{AB} + 2\eta \epsilon^{AB},$$

where  $\lambda$  and  $\eta$  are constants. This is clearly classically admissible. An obvious calculation based on (6.16) yields



$$(6.17) \quad \sigma = \sigma^{AB} h_{AB} = \sigma^{AB} k_{AB} = 3p + (2\eta + 3\lambda) \epsilon_A^A.$$

We thus obtain an immediate basis for examining the energetic analog of a general viscous fluid if we make the Stokes assumption that  $2\eta + 3\lambda = 0$ .

It must be clearly borne in mind that we are dealing with an energy continuum. The function  $p$  is thus not necessarily the relativistic analog of hydrodynamical pressure. If  $p$  were assumed to be the pressure, as in the usual discussions, certain problems arise for which there is no simple answer. Einstein clearly saw these problems and chose to follow an interpretation similar to that taken in this paper. "This [referring to  $p$ ] must not, however, be confused with a hydrodynamical pressure, as it serves only for the energetic presentations of the dynamical relations inside matter."\*

## 7. STRESS POTENTIALS

Many problems in classical continuum mechanics are significantly simplified when the material medium is such that it admits a stress potential  $\Sigma$ . The condition for this to be the case is

$$(7.1) \quad \rho \rho_0^{-1} \dot{\Sigma} = t_j^i d_i^j$$

for all  $d_i^j$ , where  $t_j^i$  is the Cauchy stress tensor,  $d_i^j$  is the rate of deformation tensor (following the motion),  $\rho$  is the mass density,  $\rho_0$  is the value of  $\rho$  in the reference state, and  $\dot{\Sigma}$  is the time derivative of  $\Sigma$  (following the motion).\*\* In the hope of achieving a similar simplification for problems in relativistic energy mechanics, we shall look for a suitable generalization of (7.1).

\*Einstein, A., The Meaning of Relativity, Third Edition, Princeton University Press, (1950), p. 106.

\*\*See C. Truesdell, "The Mechanical Foundations of Elasticity and Fluid Mechanics," J. Rational Mech. Anal., 1 (1952), p. 170.

The first problem to be resolved is that of obtaining an acceptable generalization of the left-hand side of (7.1). Since (7.1) is an equation following the motion of the classical continuum (Eulerian specification), we refer the energy continuum to a frame of reference at rest with respect to the velocity field  $W^A$  (that is,  $W^A = \delta_0^A$ ). In this frame of reference,  $x^0$  is the natural time variable of the material. Hence, differentiation with respect to  $x^0$  is the relativistic analog of the classical time differentiation following the motion. We now need to express  $x^0$  differentiation in the rest frame by means of an invariant differentiation process. Now  $W^A = \frac{dx^A}{ds}$ ; hence, in the rest frame,  $dx^0 = ds$ . We thus have established the correspondence  $\dot{\Lambda} \rightarrow \frac{d\Lambda}{ds}$ , for  $\Lambda$  a scalar function. In classical mechanics, the density  $\rho$  is the reciprocal of the element of convected volume. Thus, if  $\mathcal{S}$  denotes the slice of  $B$  over which the initial data is specified, and  $h_0$  denotes the image of  $h$  in  $\mathcal{S}$  under the inverse of the motion defined by the trajectories of the  $W^A$  field, we have the natural correspondence

$$\rho \rho_0^{-1} \rightarrow h_0 h^{-1}. \quad \text{Combining the above results, we obtain}$$

$$(7.2) \quad \rho \rho_0^{-1} \dot{\Lambda} \rightarrow h_0 h^{-1} \frac{d\Lambda}{ds}.$$

Because of the occurrence of the ratio  $h_0 h^{-1}$ , it would seem appropriate to refer to  $\Lambda$  as the convective stress potential (classically, the quantity  $\dot{\Lambda}$  should also be referred to as the convective stress potential due to the factor  $\rho \rho_0^{-1}$ ).

Turning to the right-hand side of (7.1), we note that\*

$$(7.3) \quad 2 W_{(A;B)} = \dot{\Lambda}(h_{AB}),$$

and hence

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\* See J. A. Schouten, Ricci-Calculus (Second Edition), Springer-Verlag, Berlin (1954), pp. 102-111.

$$(7.4) \quad 2 \epsilon_{AB} = \tau (h_{AB}).$$

Evaluating (7.4) in the rest frame gives\*

$$(7.5) \quad 2 \epsilon_{AB} = (h_{AB} - h_{A0} h_{B0}),_{x^0}.$$

(Remember that  $h_{00} = 1$ .) The tensor  $\epsilon_{AB}$  is thus a time derivative following the motion. The question is whether  $h_{AB} - h_{A0} h_{B0}$  may be looked upon as an Eulerian rate-of-strain tensor in the rest frame. Since  $W^A$  is a time-like unit vector, we may pick a system of four vectors  $v_b^A$  ( $b = 1, \dots, 4$ ) and a reciprocal system  $v_A^b$  such that

$$(7.6) \quad v_a^A v_B^a = \delta_B^A, \quad v_b^A v_A^a = \delta_b^a, \quad \text{and} \quad v_0^A = W^A.$$

We then have

$$(7.7) \quad h_{AB} = \beta_0 v_A^0 v_B^0 + \sum_{i=1}^3 \beta_i v_A^i v_B^i.$$

Using (7.6) and the fact that  $h_{AB} W^A W^B = 1$  gives  $\beta_0 = 1$ , and hence we have

$$(h_{AB} - h_{A0} h_{B0}) = \sum_{i=1}^3 \beta_i v_A^i v_B^i.$$

Thus, noting that  $\tau(v_A^i) = v_A^i$  and that the  $\beta_i$  are the nonholonomic components of  $h_{AB}$  in the generic space  $\mathcal{U}$  normal to  $W^A$ , we see that  $\beta_i v_A^i v_B^i$  may be looked upon as a classical strain tensor in the three-dimensional space  $\mathcal{U}$ , the exception being that the  $v_A^i$  are in general nonintegrable (i.e.,  $v_A^i \neq u_{,A}^i$ ). The tensor  $2 \epsilon_{AB}$  is thus seen to be the natural generalization of  $d_j^i$ .

In view of the above results, we have the following characterization of a convective stress potential in general relativity.

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\*Edelen, D. G. B. and T. Y. Thomas, "On the Characterization of Born and Contemporaneous Rigidity and the Question of Their Equivalence," Tensor (to be published).

**DEFINITION 6.** A scalar function  $\Lambda$  is said to be a convective stress potential of a relativistic continuum if and only if

$$(7.8) \quad h_0 h^{-1} \frac{d\Lambda}{ds} = 2 \sigma^{AB} \epsilon_{AB}$$

is an identity in  $\epsilon_{AB}$ .

The explicit occurrence of the factor  $h_0$  in (7.8) renders this equation difficult to deal with. For this reason, we transform (7.8) into an equivalent form which does not contain  $h_0$  explicitly. One may easily verify that

$$(7.9) \quad \mathcal{L}(h \exp(-\int \epsilon ds)) = 0,$$

where

$$(7.10) \quad \epsilon = \epsilon_{AB} h^{AB} = W^A_{;A}$$

and the integration is to be performed along the trajectories of the  $W^A$  field.

An integration of (7.9) thus leads to

$$(7.11) \quad h \exp(-\int \epsilon ds) = h_0.$$

In view of (7.11), we introduce the scalar density  $P$  by

$$(7.12) \quad \stackrel{\text{def}}{P} = h \Lambda \exp(-\int \epsilon ds).$$

Now  $\frac{d\Lambda}{ds} = \mathcal{L}(\Lambda)$ , and hence, by (7.9) and (7.11),

$$\mathcal{L}(P) = h \exp(-\int \epsilon ds) \mathcal{L}(\Lambda) = h_0 \frac{d\Lambda}{ds}.$$

Substituting from (7.8) into the right-hand side of the above equation, we establish the following result.

A convective stress potential  $\Lambda$  exists if and only if there exists a scalar density  $P$  such that

$$(7.13) \quad \mathcal{L}(P) = 2 h \sigma^{AB} \epsilon_{AB}$$

is an identity in  $\epsilon_{AB}$ . If this condition is satisfied, then  $\Lambda$  is given by

$$(7.14) \quad \Lambda = h^{-1} P \exp\left(\int \epsilon \, ds\right).$$

For obvious reasons, we shall refer to  $P$  as the stress potential.

## 8. STRESS POTENTIALS AND CONSTITUTIVE RELATIONS—AN EXAMPLE

It is shown in the Appendix\* that necessary and sufficient conditions for the existence of a stress potential is that the following equations be integrable for  $P$ :

$$(8.1) \quad h \sigma^{AB} = \tau(P, h_{AB}),$$

$$(8.2) \quad K + 2 A^A \dot{W}_A = 0.$$

If  $P$  exists (that is, if  $\sigma^{AB}$  satisfies the conditions (A.13)), equations (8.1) are a system of 10 constitutive relations which determine the generalized stresses.

Now, if  $P$  depends only on  $h_{AB}$  and  $\epsilon_{AB}$ , the generalized stresses given by (8.1) will be uniquely determined functions of the dynamical state of  $\mathcal{B}$ . In this case (8.2) represents an additional condition to the Einstein theory. On the other hand, if  $P$  depends on at least one other function, the relation (8.2) is consistent with the Einstein theory; in fact, it provides a natural additional constitutive relation.

We shall illustrate this result by the following example. Let

$$(8.3) \quad P = -2h(p - \lambda \epsilon);$$

then (8.1) and (8.2) give

$$(8.4) \quad \sigma^{AB} = -k^{AB}(p - \lambda \epsilon) - 2\lambda \epsilon_{CD} k^{CA} k^{BD}$$

and

$$(8.5) \quad 0 = -\mathcal{L}(p) + \epsilon \mathcal{L}(\lambda) + \lambda h^{CD} \mathcal{L}(\epsilon_{CD}),$$

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\* Apply the substitutions  $\mathcal{L} = P$  and  $\theta = 1$ , to (A.2) to obtain (7.13).

respectively now, we have

$$\epsilon(\epsilon) = h^{CD} \epsilon(\epsilon_{CD}) + \epsilon_{CD} \epsilon(h^{CD}),$$

so that (8.5) may be written in the equivalent form

$$(8.6) \quad \epsilon(p - \lambda\epsilon) = \lambda h^{AE} h^{BF} \epsilon_{AB} \epsilon(h_{EF}).$$

Also, however, we have

$$2\epsilon_{EF} = \tau(\epsilon(h_{EF})),$$

and hence a direct calculation leads to

$$h^{AE} h^{BF} \epsilon_{AB} \epsilon(h_{EF}) = 2h^{AE} h^{BF} \epsilon_{AB} \epsilon_{EF}.$$

Thus (8.5) is equivalent to

$$(8.7) \quad \epsilon(p - \lambda\epsilon) = 2\lambda h^{AE} h^{BF} \epsilon_{AB} \epsilon_{EF}.$$

Since  $p - \lambda\epsilon$  is a scalar, integration of (8.7) leads to

$$(8.8) \quad p = p_0 - \lambda_0 \epsilon_0 + \lambda\epsilon + 2 \int \lambda h^{AE} h^{BF} \epsilon_{AB} \epsilon_{EF} ds,$$

where the integration is to be performed along the trajectory of the  $W^A$  field from the appropriate point in  $\mathcal{I} \cap \mathcal{B}$  to the point in  $\mathcal{B}$  at which the left-hand side of (8.8) is to be evaluated. (Here,  $p_0$ ,  $\lambda_0$ , and  $\epsilon_0$  are the values of  $p$ ,  $\lambda$ , and  $\epsilon$  in  $\mathcal{I} \cap \mathcal{B}$ .) Hence, for given  $\lambda$ , both the generalized stresses  $\sigma^{AB}$  and the function  $p$  are uniquely determined functions of the dynamical state of  $\mathcal{B}$ .

On the other hand, if we were to set  $p = 0$  and consider  $\lambda$  as given, the above analysis would yield

$$(8.9) \quad \sigma^{AB} = \lambda \epsilon^k{}^{AB} - 2\lambda \epsilon_{CD}^k{}^{CA}{}^k{}^{BD}$$

and

$$(8.10) \quad \frac{d(\lambda\epsilon)}{ds} = 2\lambda h^{AE} h^{BF} \epsilon_{AB} \epsilon_{EF} \quad (0 = \epsilon(\lambda\epsilon)).$$

In this case we again obtain a determination of the generalized stresses, but only for those motions which are such that (8.10) is satisfied.

Consider now the case for which  $\lambda = 0$ . This gives

$$(8.11) \quad P = -2 h p, \quad \sigma^{AB} = -p k^{AB},$$

and

$$(8.12) \quad \mathcal{L}(p) = 0.$$

With  $\sigma^{AB}$  given by (8.11), the basic equation (3.16) leads to

$$(8.13) \quad \mathcal{L}(h\mu) = -p \mathcal{L}(h).$$

Thus, applying the result (8.12) we have the following conclusion. The existence of a stress potential for  $-p k^{AB}$  implies the conservation law

$$(8.14) \quad (\rho W^A)_{;A} = 0, \quad \rho = \mu + p.$$

Combining this conclusion with the results of Thomas\* we see that a region  $\mathcal{B}$  of  $\mathcal{E}$  with the material momentum-energy tensor

$$T_{AB} = \rho W^A W^B - p h^{AB}, \quad \rho = \mu + p > 0$$

contains a material point whose trajectory is a geodesic if the stress tensor  $-p k^{AB}$  admits a stress potential.

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\* T. Y. Thomas, "On the Geodesic Hypothesis in the Theory of Gravitation," Proc. Nat. Acad. Sci. (U.S.A.), 48 (1962), pp. 1567-1569.

## 9. PATH DENSITY

The statement of a material momentum-energy tensor of class one is seen to require the statement of the generalized stress tensor  $\sigma^{AB}$ . In general, the stress tensor characterizes the resistance of the energy continuum  $\mathcal{B}$  to deformations; as such, this tensor depends on the particular properties of the energy continuum under consideration. Statement of  $\sigma^{AB}$  is thus not an appended condition to the Einstein theory; it serves to characterize the actual properties of the particular energy continuum under investigation.

We shall henceforth assume that the  $\sigma^{AB}$  are known functions of the state of motion of  $\mathcal{B}$ . Since  $\sigma^{AB}$  has the generic three-dimensional space  $\mathcal{U}$  as support,\* this tensor will, in general, depend on  $k_{AB}$ ,  $\pi_A^B$ ,  $k^{AB}$ , and  $\epsilon_{AB}$  as well as any other fields which are considered in the particular problem under investigation. Hence, under solution of the Einstein equations, the tensors  $\sigma^{AB}$  and  $\epsilon_{AB}$  will be known functions along the trajectories of the  $W^A$  field.

The state of motion of an energy continuum  $\mathcal{B}$  is determined by solving the Einstein field equations together with the basic equations of energy mechanics given in Sec. 3. In order that this may be accomplished, appropriate initial (Cauchy) data must be specified. Let  $\mathcal{S}$  be the section of  $\mathcal{B}$  over which the initial data is specified. In addition, let  $\xi$  be the assigned value of  $\mu$  on  $\mathcal{S} \cap \mathcal{B}$ .

A function fundamental in our consideration is the path density  $S$ . This function is a scalar density defined on the trajectories of the  $W^A$  field by the conditions

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\*The generic space  $\mathcal{U}$  is unique owing to the fact that the vector field  $W^A$  is unique.



$$(9.1) \quad f(S) = h \sigma^{AB} \epsilon_{AB}, \quad h = (-\det(h_{AB}))^{1/2},$$

and

$$(9.2) \quad S|_{\int \cap \theta} = 0.$$

By definition of the Lie derivative of a scalar density, we have

$$(9.3) \quad f(S) = \frac{dS}{ds} + S W^A_{;A} = \frac{dS}{ds} + S \epsilon,$$

where  $s$  is the arc-length parameter along the trajectories of the  $W^A$  field and

$$(9.4) \quad \epsilon = \epsilon_{AB} h^{AB} = \epsilon_{AB} k^{AB}.$$

Substituting (9.3) into (9.1) and using the condition (9.2) to evaluate the arbitrary function which arises in the integration process, we obtain the following explicit evaluation of the path density:

$$(9.5) \quad S = \exp(-\int \epsilon ds) \int \exp(\int \epsilon ds) h \sigma^{AB} \epsilon_{AB} ds,$$

the integrations being performed along the trajectories of the  $W^A$  field from the appropriate point in  $\int \cap \theta$  to the point under consideration. Since the generalized stresses are assumed to be known on the trajectories of the  $W^A$  field, (9.5) determines a unique function on the trajectories of  $W^A$ .

There are several transformations of the path density  $S$  which will be useful. If we set

$$(9.6) \quad S = hQ,$$

then  $Q$  is a scalar function and (9.1) becomes

$$(9.7) \quad f(Q) = \sigma^{AB} \epsilon_{AB}.$$

Proceeding in the same manner as above, we then obtain

$$(9.8) \quad Q = \exp(-\int \epsilon ds) \int \exp(\int \epsilon ds) \sigma^{AB} \epsilon_{AB} ds.$$

The function  $Q$  will be referred to as the path function. If we set

$$(9.9) \quad S = q \exp(-\int \epsilon ds),$$

then  $q$  is a scalar function. By the identity

$$(9.10) \quad f(h \exp(-\int \epsilon ds)) = 0,$$

equation (9.1) becomes

$$(9.11) \quad \exp(-\int \epsilon ds) f(q) = \sigma^{AB} \epsilon_{AB}.$$

Integrating (9.11), we then obtain

$$(9.12) \quad q = \int \exp(\int \epsilon ds) \sigma^{AB} \epsilon_{AB} ds.$$

This function will be referred to as the path kernel.

## 10. CONSERVED QUANTITIES

Knowledge of the path density  $S$  allows us to prove the following fundamental result.

The Einstein field equations imply the existence and uniqueness of the scalar density

$$(10.1) \quad \dagger \stackrel{\text{def}}{=} h \mu - S$$

such that

$$(10.2) \quad f(\dagger) = (\dagger W^A)_{,A} = 0.$$

The proof is as follows. Let  $S$  be given by (9.5); then (9.1) holds.

By (3.16), the Einstein field equations imply

$$(10.3) \quad f(h \mu) = \sigma^{AB} \epsilon_{AB}.$$

Eliminating the expression  $\sigma^{AB} \epsilon_{AB}$  between (9.1) and (10.3) gives

$$(10.4) \quad \mathcal{L}(\mu h - S) = 0.$$

Now,  $\mu$  is an eigen value of  $T^{AB}$ , so that  $k\mu + \frac{1}{2}R$  is an eigen value of the Ricci Tenson.\* Hence  $\mu$  is a differential invariant of the Einstein-Riemann space under consideration. The density  $\mu h$  is thus uniquely determined by the geometry of the space-time continuum. The path density  $S$  is also unique. Hence the scalar density  $\dot{\phi} = \mu h - S$  is unique and satisfies (10.2).

There are several transformations of  $\dot{\phi}$  which will be useful. If we set  $\dot{\phi} = h\varphi$ , then  $\varphi$  is a scalar function such that

$$(10.5) \quad 0 = (\varphi W^A)_{;A} = \frac{d\varphi}{ds} + \epsilon \varphi = \mathcal{L}(\varphi) + \epsilon \varphi$$

and  $\varphi = \mu - Q$ . In obtaining the last result we have, of course, used (9.6). If we set  $\dot{\phi} = \psi h \exp(-\int \epsilon ds)$ , then  $\psi$  is a scalar function. By use of the identity (9.10), equation (10.2) reduces to  $0 = \mathcal{L}(\psi) = \frac{d\psi}{ds}$ . Hence  $\psi$  is always a constant of the motion. On the other hand,  $\varphi$  is a constant of the motion only if the motion is incompressible, that is  $\epsilon = 0$ . In an obvious manner, we also obtain  $\psi = \mu \exp(-\int \epsilon ds) - q$ , when use is made of (9.9).

## 11. THE EXISTENCE AND UNIQUENESS OF INTRINSIC ENERGY

As a consequence of Einstein's fundamental discovery that mass and energy are equivalent, any change in the energy of a body results in a change in its mass. Since the energy of a body is a function of its dynamical state, so is its mass. Fundamental to the idea of a material body, however, is the existence of an intrinsic mass (or energy) which is immutable. Accordingly, when discussing material

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\*This is an immediate consequence of the Einstein field equation  $R_{AB} - R h_{AB}/2 = k T_{AB}$ .

bodies in the general theory of relativity, additional conditions are often imposed so as to secure an intrinsic mass. This section shows that it is unnecessary to impose such additional conditions - the Einstein field equations imply the existence and uniqueness of just the required intrinsic quantity.

We have shown that if a region  $\mathcal{B}$  of an Einstein-Riemann space is occupied by a material body (that is, the associated momentum-energy tensor of  $\mathcal{B}$  is material and of class one), then there exists a unique scalar density  $\dagger$  on  $\mathcal{B}$  such that

$$(11.1) \quad (\dagger W^A)_{,A} = 0$$

and

$$(11.2) \quad h\mu = \dagger + S.$$

Equation (11.1) states that the flux of  $\dagger$  is conserved during the motion of the body. Since  $h\mu$  is the density of rest-energy (that is, the energy which would be observed in a coordinate system such that  $W^A = \delta^A_0$ ), equation (11.2) states that the density of rest energy is uniquely representable as the sum of a path-dependent density  $S$  (the path density) and a density  $\dagger$  which is conserved. Now, by (9.2) and (11.2), we have

$$h \left| \int_{\partial \mathcal{B}} \mu \right|_{\partial \mathcal{B}} = h \left| \int_{\partial \mathcal{B}} S \right|_{\partial \mathcal{B}} + \dagger \left| \int_{\partial \mathcal{B}} \right|$$

Thus,  $\dagger$  is a density of energy. Since  $\dagger$  is conserved, it is intrinsically associated with the body. Hence, the Einstein theory implies the existence and uniqueness of an intrinsic energy density of a material body, namely  $\dagger$ .

The functions  $\omega$  and  $\dagger$  introduced in the last section may now be interpreted. Since

$$(11.3) \quad \phi = u - Q, \quad \phi \Big|_{\partial B} = u \Big|_{\partial B} = \xi,$$

and

$$(11.4) \quad (\phi W^A)_{;A} = 0,$$

we may identify  $\phi$  with the intrinsic rest-energy. Noting that  $u$  is the total rest-energy, we see that  $Q$  is the expendable rest-energy; namely, that part of the rest-energy which is a consequence of the dynamic processes to which the body is subject. From the identity

$$(11.5) \quad \mathcal{L}(h \exp(-\int \epsilon ds)) = 0,$$

we see that  $\exp(\int \epsilon ds)$  is proportional to the convected element of geometrical volume; that is, we have  $\exp(\int \epsilon ds) = h/h_0$  where  $h_0$  is the value of  $h$  at the corresponding point in  $\mathcal{V}$ . Thus since  $\psi = u \exp(-\int \epsilon ds) - q$  and  $\frac{d\psi}{ds} = 0$ , we see that  $\psi$  is the intrinsic rest-energy per unit of convected geometrical volume. In particular, we have  $h_0^{-1}(uh - \xi h_0) = q$ , so that  $q$  is the total change in rest-energy density per unit of initial geometrical volume.

## 12. THE QUESTION OF PATH DEPENDENCE

We have seen that the path density  $S$  is given by

$$(12.1) \quad S = \exp(-\int \epsilon ds) \int \exp(\int \epsilon ds) h \sigma^{AB} \epsilon_{AB} ds,$$

and that this quantity together with  $hu$  serves to determine the intrinsic energy of an energy continuum. In general, the path density will depend directly on the trajectories of the  $W^A$  field in addition to the quantities appearing in the integrands of (12.1). In this section we examine the conditions under which  $S$  is path independent in an appropriate sense.

If  $S$  is to be path independent, we must require the integrand

$\exp(\int \epsilon ds) h \sigma^{AB} \epsilon_{AB}$  to be a total differential. We thus suppose that the generalized stress field is such that there exists a scalar function  $\Lambda$  on  $\mathcal{B}$  with the property that

$$(12.2) \quad \frac{d}{ds} (h \exp(\int \epsilon ds) \Lambda) = h \exp(\int \epsilon ds) \sigma^{AB} \epsilon_{AB}$$

holds for all kinematically possible motions in  $\mathcal{B}$ . If this supposition is satisfied,

(12.1) gives

$$(12.3) \quad S = h\Lambda - \exp(-\int \epsilon ds) h_0 \Lambda_0,$$

where  $h_0$  and  $\Lambda_0$  are the values of  $h$  and  $\Lambda$  at the corresponding point in  $\mathcal{B} \cap \mathcal{B}'$ .

We thus have to examine the conditions under which (12.2) holds for all kinematically possible motions in  $\mathcal{B}$ . Now,

$$\frac{d}{ds} [h \exp(\int \epsilon ds) \Lambda] = h \exp(\int \epsilon ds) \left[ \epsilon \Lambda + \frac{d\Lambda}{ds} \right],$$

and hence (12.2) is equivalent to

$$(12.4) \quad \sigma^{AB} \epsilon_{AB} = \frac{d\Lambda}{ds} + \epsilon \Lambda.$$

Noting that  $\mathcal{L}(h) = \epsilon h$ , we finally obtain the condition

$$(12.5) \quad \mathcal{L}(\Lambda h) = h \sigma^{AB} \epsilon_{AB}$$

for all kinematically admissible  $\epsilon_{AB}$ . This, however, is exactly the requirement for the existence of a stress potential  $P$ . Hence, setting  $P = 2h\Lambda$ , we have the following result: A necessary and sufficient condition for  $S$  to be path independent in the above sense is that there exists a stress potential  $P$ . If this condition is satisfied, then

$$(12.6) \quad S = P/2 - \exp(-\int \epsilon ds) P_0/2.$$

We have seen that in the case of a path density, we have the intrinsic density function

$$(12.7) \quad \dot{\phi} = h\dot{u} - S.$$

Under the above result, (12.6) and (12.7) give

$$(12.8) \quad \dot{\phi} = h\dot{u} - P/2 + \exp(-\int \epsilon ds) P_0/2$$

However,  $\delta(\exp(-\int \epsilon ds) P_0/2) \equiv 0$ , and hence the quantity  $h\dot{u} - P/2$  is an intrinsic quantity. In addition, since  $P/2$  is path independent and hence a function of position only, we have the local conservation law

$$(12.9) \quad \{ (h\dot{u} - P/2) W^A \}_{,A} = 0$$

at all points in  $\mathcal{B}$ . We thus see that path independence of the path function implies the existence of a local conservation law.

### 13. A FUNDAMENTAL DIFFERENTIAL ON THE TRAJECTORIES OF $W^A$ (THE LAW OF THERMODYNAMICS).

As a consequence of the Einstein theory and the requirement that a region  $\mathcal{B}$  have an associated momentum-energy tensor which is material and of class one, the dynamical processes interior to  $\mathcal{B}$  can always be viewed as an energy mechanics. In the customary approach, any basic energy mechanics is governed by the laws of thermodynamics. One is thus led to inquire as to the relations between the previous results and the laws of thermodynamics.

Although it may have seemed disguised, we have already derived the first law. One of the basic equations of relativistic energy mechanics reads

$$(13.1) \quad \mathcal{L}(h\dot{u}) = h \sigma^{AB} \epsilon_{AB}.$$

Since the Lie derivative formed from  $W^A$  is the relativistic analog of the time derivative following the motion, equation (13.1) states that the time derivative of the rest-energy density following the motion is equal to the power developed by the generalized stresses per unit of geometrical volume  $dx^1 dx^2 dx^3 dx^4$  - the first law.

In order to proceed further, we must use the results of the Appendix to decompose the generalized stress tensor. Set

$$(13.2) \quad \sigma^{AB} = L^{AB} + M^{AB} + N^{AB},$$

where  $L^{AB}$  and  $M^{AB}$  are those parts of  $\sigma^{AB}$  for which

$$(13.3) \quad \mathfrak{L}(P) = 2 h L^{AB} \epsilon_{AB}$$

$$(13.4) \quad \mathfrak{L}(\Sigma) = 2 \theta h M^{AB} \epsilon_{AB}$$

are identities in  $\epsilon_{AB}$ , and  $N^{AB}$  is what is left. Thus,  $L^{AB}$  satisfies  $E^{ABCD}(L) = 0$ , and  $M^{AB}$  satisfies  $E^{ABCD}(M) M^{FG} + E^{CDFG}(M) M^{AB} + E^{FGAB}(M) M^{CD} = 0$  where  $E^{ABCE}(\cdot)$  is the operator defined by (A. 12).

Now, if we substitute (13.2) into the right-hand side of (13.1) and use (13.4), we are led to the expression

$$(13.5) \quad \mathfrak{L}(hu) = \mathfrak{L}(P/2) + \theta^{-1} \mathfrak{L}(\Sigma/2) + \mathfrak{L}$$

where

$$(13.6) \quad \mathfrak{L} = h N^{AB} \epsilon_{AB}.$$

By definition,  $D(B) \equiv \mathfrak{L}(B) ds$  is the Lie differential of any geometric object  $B$  with respect to the trajectories of the  $W^A$  field, and hence is a differential following the motion - a convective differential. Thus, equation (13.5) leads to the following fundamental differential on the trajectories of  $W^A$ :

$$(13.7) \quad D(hu) = D(P/2) + \theta^{-1} D(\Sigma/2) + \mathfrak{L} ds.$$

Associated with this we have the intrinsic quantity

$$(13.8) \quad \mathfrak{L} = hu - P/2 + \exp(-\int \epsilon ds) P_0/2 - \int \mathfrak{L} ds - \frac{1}{2} \exp(-\int \epsilon ds) \int \theta^{-1} d(\Sigma \exp(\int \epsilon ds)).$$

We now have the desired result. Equation (13.7) states that the convective differential of the density of rest-energy is equal to the convective differential of



the recoverable energy stored in the deformations of the energy continuum (i. e. , the energy represented by the path independent potential  $P/2$ ) plus the irrecoverable energy increments represented by the convective differential form  $\theta^{-1} D(\mathcal{E}/2)$  and  $\sharp ds$ . The convective differential form  $\theta^{-1} D(\mathcal{E}/2)$  is, however, exactly analogous to that encountered in classical thermodynamics - the convective differential operator  $D$  being that which is used in proper formulations of thermodynamics. We are thus led in a natural fashion to define  $\theta^{-1}$  as the intrinsic temperature and  $\mathcal{E}/2$  as the intrinsic entropy density of the energy continuum. In fact, if we insist that

$$(13.9) \quad D(\mathcal{E}/2) \geq 0 ,$$

then (13.7) and (13.8) state the relativistic analogs of the first and second laws.

A general material region of an Einstein-Riemann space will not necessarily be such that (13.9) is satisfied. The relativistic energy mechanics is thus a richer subject than that which would be obtained through a purely thermodynamic description. This added richness is, however, no blessing in disguise - if anything, relativistic mechanics is all too rich for our analytical blood. We therefore take the opportunity afforded by the above result to constrain the universe of discourse of our mechanics.

**DEFINITION 7.** A material region of an Einstein-Riemann space is said to be thermodynamically admissible if and only if  $D(\mathcal{E}/2) \geq 0$ .

The differential form  $\sharp ds$  may cause some pause to those accustomed to the classical notations of thermodynamics, to say nothing of the fact that  $D(h\mu)$  is generally not an exact form. The inexactness of the form  $D(h\mu)$  is easily dispensed with on noting that, contrary to classical notions, the mass of a

relativistic body depends intrinsically on its history, and hence the system is not locally closed with respect to the measure  $\mu$ . If one desires, the existence of intrinsic energy can be used to reformulate the description in terms of a system which is closed with respect to its history. At this point in our work we can only conjecture the significance of the form  $\psi ds$ . If  $\psi = 0$ , (13.7) gives  $D(hu) = D(P/2) + \theta^{-1} D(E/2)$ . Now,  $D(hu) = h \sigma^{AB} \epsilon_{AB} ds$  by (3.16), so that the above equation becomes

$$(13.10) \quad D(P/2) = h \sigma^{AB} \epsilon_{AB} ds - \theta^{-1} D(E/2).$$

In this equation,  $D(P/2)$  is an exact form while  $\sigma^{AB} \epsilon_{AB} ds$  has the dimensions of work. Hence (13.10) gives a description of  $\mathcal{B}$  which is locally closed with respect to  $P/2$  and which has the familiar form " $dW - T dS$ ". If, however, a system emits gravitational waves, it should be impossible to obtain a locally closed system with just the usual thermodynamic variables. Hence we conjecture that the differential form  $\psi ds$  describes the thermodynamic effects of gravitational radiation.

#### 14. AN EXAMPLE

The definitions of intrinsic temperature and intrinsic entropy density given in the last section are of a formal nature. In order to clothe this formality with physical significance, we examine these concepts in the context of a specific problem.

Consider a material region  $\mathcal{B}$  of an Einstein-Riemann space with generalized stress tensor given by

$$(14.1) \quad \sigma^{AB} = -p k^{AB},$$

where  $p$  is a scalar function. It has been pointed out that the stress tensor

(14.1) admits a stress potential if and only if  $\mathcal{L}(p) = \frac{dp}{ds} = 0$ . If we substitute

(14.1) into (3.16), we have

$$(14.2) \quad \frac{dp}{ds} = h^{-1} \mathcal{L}(h\rho) \stackrel{\text{def}}{=} \chi,$$

where  $\rho = \mu + p$  and  $h\rho$  is what is usually identified with the mass density of the material region.

If the conclusions of the last sections are correct, we should be able to obtain all significant thermodynamical variables as consequences of the Einstein theory. Let  $'p$  be defined by  $'p = \int \chi ds$ , where  $\chi$  is given by  $h^{-1} \mathcal{L}(h\rho)$  and the integration is to be performed along the trajectories of the  $W^A$  field. We then define the scalar  $''p$  by  $''p = p - 'p$ , so that  $p = 'p + ''p$  and  $\frac{d''p}{ds} = 0$ . The decomposition (13.2) gives

$$(14.3) \quad \sigma^{AB} = -('p + ''p) k^{AB},$$

from which it may easily be verified that the equation

$$(14.4) \quad \mathcal{L}(P) = -2 h('p k^{AB})_{;AB}$$

is satisfied for all kinematically admissible  $\epsilon_{AB}$  with  $P = -2 h 'p$ . Hence,

by (13.3) and the above result, we have  $L^{AB} = -''p k^{AB}$  and  $M^{AB} = -'p k^{AB}$ ,  $\frac{d'p}{ds} = q$ , it being assumed that  $\psi = 0$ . We therefore look for functions  $\theta$  and

$\Sigma$  such that (13.4) is satisfied. This leads to the relation

$$(14.5) \quad \mathcal{L}(\Sigma) = 2 h \epsilon M^{AB} \epsilon_{AB} = -2 h \epsilon 'p \epsilon.$$

Since  $\mathcal{L}(h) = h\epsilon$ , the most general integral of (14.5) is

$$(14.6) \quad \Sigma = -2 h \theta 'p + U,$$

where

$$(14.7) \quad 2 h \mathcal{L}('p \theta) = \mathcal{L}(U);$$

the latter equation being nothing more than the condition stated by (A.8).

Equations (14.6) and (14.7) are two relations between the three unknowns  $\epsilon$ ,  $\Sigma$ , and  $U$ . We must therefore have one additional relation. Now, as previously noted, the stress tensor (14.1) requires the specification of one additional constitutive relation in view of the occurrence of the function  $p$ . This is exactly reflected by the need for one additional relation for the determination of  $\epsilon$ ,  $\Sigma$ , and  $U$ . We thus see that we may prescribe the function  $U$ , and thereby prescribe the required constitutive relation. This amounts to prescribing the equation of state for the material. In the simplest case, namely  $U = 0$ , (14.7) leads to  $p = p_0 \theta_0$ , which may be written in the more suggestive form

$$(14.8) \quad \frac{p}{T} = \frac{p_0}{T_0},$$

where the definition of intrinsic temperature is used:  $T = \frac{1}{\theta}$  def intrinsic temperature. If we take

$$(14.9) \quad \mathcal{L}\left(\frac{U}{2}\right) = \frac{\lambda h \exp(-\int \epsilon ds)}{p} \left[ \frac{p}{p} \mathcal{L}(p) - 'p \epsilon \right],$$

where  $\frac{d\lambda}{ds} = 0$ , then (14.7) yields  $p \theta = \lambda \exp(-\int \epsilon ds)$ . Rewriting this with  $T$  instead of  $\theta^{-1}$  and multiplying by  $h$  gives  $\frac{hp}{T} = \lambda h \exp(-\int \epsilon ds)$ , so that we have  $\mathcal{L}\left(\frac{hp}{T}\right) = 0$ , the state equation for a "perfect fluid." From the above examples, we see that not only do the ideas introduced in the last section make sense for the case of a general gas; they also provide a natural manner of stating relativistically valid equations of state.

As a last note, we examine the implication of the requirement that the region be thermodynamically admissible. From (14.5) we have  $\mathcal{L}(\Sigma) = -2 h \theta 'p \epsilon$ ,

and hence the requirement (13.9) reads  $h \theta 'p \leq 0$ . Now,  $h$  is strictly positive and  $\theta$  may be taken as strictly positive since the reference value of  $\theta$  is undetermined by (14.7). Hence, (14.1) describes a thermodynamically admissible region if and only if  $'p \leq 0$ ; that is, if and only if a positive "pressure" can only produce a negative dilatation.

#### APPENDIX. EXISTENCE REQUIREMENTS

Let the symmetric tensor  $\sigma^{AB}$  be given and such that

$$(A.1) \quad \pi(\sigma^{AB}) = \sigma^{AB}.$$

We examine under what conditions a scalar  $\theta$  and a scalar density  $\Sigma$  exist such that

$$(A.2) \quad \mathcal{L}(\Sigma) = 2 \theta h \sigma^{AB} \epsilon_{AB},$$

is an identity in the Born rate-of-strain tensor

$$(A.3) \quad \epsilon_{AB} = \pi(W_{(A;B)}) = \pi(\mathcal{L} h_{AB})/2$$

Set

$$(A.4) \quad \mathcal{L}(\Sigma) = \Sigma,_{h_{AB}} \mathcal{L}(h_{AB}) + K$$

where  $K$  is defined for any  $P$  be formal expansion of the left-hand side of (A.4).

If we decompose the product in (A.4) with the projection operator  $\pi$ , we have

$$(A.5) \quad \mathcal{L}(\Sigma) = \pi(\Sigma,_{h_{AB}}) \cdot \pi(\mathcal{L} h_{AB}) + K + (1 - \pi)(\Sigma,_{h_{AB}}) \cdot (1 - \pi)(\mathcal{L} h_{AB}),$$

so that (A.2) and (A.3) lead to

$$(A.6) \quad 2 \{ \pi(\Sigma,_{h_{AB}}) - \theta h \sigma^{AB} \} \epsilon_{AB} + K + (1 - \pi)(\Sigma,_{h_{AB}}) \cdot (1 - \pi)(\mathcal{L} h_{AB}) = 0.$$

If (A.6) is to be an identity in  $\epsilon_{AB}$ , it must hold for Born rigid motions (that is,

motions for which  $\epsilon_{AB} = 0$ ). Hence we obtain the condition

$$(A.7) \quad K + (1 - \pi)(\mathcal{L}_{h_{AB}}) \cdot (1 - \pi)(\mathcal{L}_{h_{AB}}) = 0.$$

Now,

$$(1 - \pi)(\mathcal{L}_{h_{AB}}) = 2 W_A \dot{W}_B,$$

while  $(1 - \pi)(\mathcal{L}_{h_{AB}})$  admits the representation

$$(1 - \pi)(\mathcal{L}_{h_{AB}}) = 2 A^A W^B + B W^A W^B$$

with

$$B = W_A W_B (1 - \pi)(\mathcal{L}_{h_{AB}}), \quad A^A = W_B (1 - \pi)(\mathcal{L}_{h_{AB}}) - B W^A.$$

Hence we have

$$(1 - \pi)(\mathcal{L}_{h_{AB}}) \cdot (1 - \pi)(\mathcal{L}_{h_{AB}}) = 2 A^A \dot{W}_A$$

on noting that  $W^B \dot{W}_B = 0$  as a consequence of  $W_A W^A = 1$ . Combining this with the above result shows that a necessary condition for the existence of  $\theta$  and  $\mathcal{L}$  is

$$(A.8) \quad K + 2 A^A \dot{W}_A = 0.$$

If this condition is satisfied, (A.1) and (A.6) imply that\*

$$(A.9) \quad \theta h \sigma^{AB} = \pi(\mathcal{L}_{h_{AB}}).$$

Conversely, if  $\sigma^{AB}$  is defined in terms of  $\mathcal{L}$  and  $\theta$  by (A.9) and (A.8) is satisfied, then (A.4) reduces to an identity in  $\epsilon_{AB}$ . Hence, a necessary and sufficient condition for the existence of functions  $\theta$  and  $\mathcal{L}$  such that (A.2) be an identity in  $\epsilon_{AB}$  is that (A.8) and (A.9) be integrable for  $\mathcal{L}$ .

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\* Since the rank  $(\epsilon_{AB}) \leq 3$ , (A.6) implies  $\theta h \sigma^{AB} = \pi(\mathcal{L}_{h_{AB}}) + M^{AB}$ , where  $M^{AB} \epsilon_{AB} = 0$ . The condition (A.1) together with the fact that (A.6) is to be an identity in  $\epsilon_{AB}$  and hence must hold for  $\epsilon_{AB}$  of maximum rank, then yields  $M^{AB} = 0$ .

In view of the above results, we would like to establish conditions under which (A. 9) holds. The first thing we must do is to bring the right-hand side of (A. 9) into a more tractable form. From the definition of  $k_{AB}$ , namely  $k_{AB} = \pi(h_{AB})$ , we have  $h_{AB} = k_{AB} + W_A W_B$ . Hence, since the projectors  $\pi_A^B = \delta_A^B - W_A W^B$  are idempotent and  $k_{AB}$  is a projection, we have

$$h_{AB} = \pi_A^C \pi_B^D k_{CD} + W_A W_B,$$

which describes the projective decomposition of  $h_{AB}$ .<sup>\*</sup> From this equation we formally obtain

$$h_{AB}, k_{CD} = \pi_A^C \pi_B^D.$$

Hence, if we replace all functions of  $h_{AB}$  appearing in  $\mathcal{E}$  by  $k_{AB} + W_A W_B$ , we have

$$\mathcal{E}, k_{AB} = \mathcal{E}, h_{CD} h_{CD}, k_{AB} = \pi_A^C \pi_B^D \mathcal{E}, h_{CD} = \pi(\mathcal{E}, h_{AB}).$$

The system (A. 9) may thus be written in the equivalent form

$$(A. 10) \quad \theta h \sigma^{AB} = \mathcal{E}, k_{AB}.$$

The integrability conditions for such systems are well known,\*\* and all that must be done is to translate these conditions into a form appropriate to the problem at hand. Define the quantities  $E^{ABCD}(o)$  as follows:

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<sup>\*</sup>This is seen from the fact that  $h_{AB} = \pi_A^C \pi_B^D G_{CD} + W_A W_B$  gives a representation of  $h_{AB}$  for any symmetric, nonsingular tensor  $G_{AB}$ .

<sup>\*\*</sup>Forsyth, A. R.: Theory of Differential Equations, Vol. 1, Dover Publications, Inc., New York (1959), p. 12.

$$(A. 11) \quad E^{ABCD}{}_{(c)} = (h \sigma^{AB})_{,k_{CD}} - (h \sigma^{CD})_{,k_{AB}}$$

We then have the following result. Necessary and sufficient conditions for the existence of functions  $\theta$  and  $E$  that satisfy (A. 10) are

$$(A. 12) \quad E^{ABCD}{}_{\sigma}{}^{FG} + E^{CDFG}{}_{\sigma}{}^{AB} + E^{FGAB}{}_{\sigma}{}^{CD} = 0.$$

It is also to be noted that the above conditions reduce to

$$(A. 13) \quad E^{ABCD} = 0$$

for the case  $\theta = 1$ .